

New three-wave-hybrid solutions for the (3+1)-dimensional generalized shallow water wave equation

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Abstract

This paper presents **novel three-wave-hybrid solutions** for the (3+1)D generalized shallow water wave equation, derived via the **combined application** of the Hirota bilinear method and the test function approach. These solutions introduce an arbitrary function and a multitude of arbitrary constants, enriching the diversity of solutions and offering a more comprehensive view of the solution's structural complexity. The unveiled nonlinear phenomena within the equation provide deeper insights into the underlying dynamics, which are crucial for understanding complex wave dynamics within both aquatic and atmospheric environments. The solutions have significant implications for the fields of hydrodynamics, geophysics, and atmospheric science, as they can be used to study the propagation and attenuation of waves in shallow water and the effects of natural phenomena such as tides and storms. Digital images accompanying the study partially illustrate the propagation patterns of the waves, enhancing the visual comprehension of the theoretical findings and providing a clearer understanding of the practical applications in predicting and managing wave behavior.

Keywords: The (3+1)D generalized shallow water wave equation; The bilinear form; The three-wave-hybrid solution

1 Introduction

In the realm of nonlinear science, the (3+1)D generalized shallow water wave equation stands as a cornerstone for understanding complex wave dynamics within both aquatic and atmospheric environments. This fundamental mathematical model is characterized by the following expression[1-2]:

$$u_{yt} + u_{xxx}y - 3u_xu_{xy} - 3u_{xx}u_y - u_{xz} = 0,$$

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This model **provides a critical framework** for describing wave phenomena in oceanic and atmospheric systems, enabling the study of wave propagation and attenuation in shallow water as well as the effects of natural events such as tides and storms[3]. Therefore, it has great significance and value in the fields of hydrodynamics, geophysics and atmospheric science. Many researchers have studied this equation and obtained innovative results which include the structures and propagation behavior of the solutions [1-21], such as multiple-soliton solutions[3], solitary wave solutions and periodic wave solutions[6,7,12,13], lump solutions and rogue wave solutions[9-11] , periodic solitary wave solutions[8], Grammian and Pfaffian solutions[15], traveling wave solutions and non-traveling wave solutions[16], rational solutions and lump solutions[18]; Some researchers have also studied the variable-coefficient (3+1)-dimensional generalized shallow water wave equation and have obtained soliton solutions and periodic wave solutions[7]. Among the results obtained, the Hirota bilinear method is the most applied approach. Despite the solutions of various structures have been obtained for the the(3+1)D generalized shallow water wave equation, however, we also find that the solution of this equation still has new structures and evolution.

In this work, we have employed the Hirota bilinear method and the test function method to derive two innovative three-wave hybrid solutions for Eq.(1)[23-26]. These solutions are characterized by the presence of an arbitrary function and numerous arbitrary constants, which allow for the generation of solutions exhibiting diverse characteristics, such as chain soliton solution, periodic three- wave solution and lump solution. This breakthrough provides deeper insights into the nonlinear mechanisms embedded within Eq.(1), thereby enhancing our comprehension of natural phenomena and facilitating the development of more efficacious strategies for control and optimization.

The paper is organized as follows: Section 2 outlines the derivation of the bilinear form for Eq. (1) and presents several exact solutions. Section 3 offers a partial analysis of the dynamical behavior of these solutions. Finally, Section 4 summarizes the findings and conclusions of the study.

2 The bilinear form and three-wave-hybrid solutions for Eq.(1)

First, we suppose

$$u(x, y, z, t) = -2\partial_x[\ln(f(x, y, z, t))], \quad (2)$$

Substitute Eq. (2) into Eq. (1) yields:

$$-2\partial_x[\ln(f)_{yt}] - 2\partial_x[\ln(f)_{xxxy}] - 12\partial_x[\ln(f)_{xx}\ln(f)_{xy}] + 2\partial_x[\ln(f)_{xz}] = 0, \quad (3)$$

Integrating Eq. (3) with respect to x (setting the integration constant to zero) and simplifying, we obtain:

$$\frac{2(f_{yt}f - f_y f_t) + 2(f_{xxxy}f + 3f_{xx}f_{xy} - 3f_{xy}f_{xx} - f_{xxx}f_y) - 2(f_{xz}f - f_x f_z)}{f^2} = 0, \quad (4)$$

or equivalent bilinear equation

$$(D_y D_t + D_x^3 D_y - D_x D_z)f \cdot f = 0, \quad (5)$$

where the operator D_x, D_y, D_z, D_t are original Hirota bilinear operators[18], which are defined by

$$D_x^m D_y^n D_z^i D_t^j (f \cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^i \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^j f(x, y, z, t) g(x', y', z', t') | x' = x, y' = y, z' = z, t' = t. \quad (6)$$

where $f(x, y, z, t)$ is a differentiable function of x, y, z and t , $g(x', y', z', t')$ is a differentiable function of the formal variables x', y', z' and t' , and m, n, i and j are the non-negative integers.

Secondly, two test functions of three-wave structure are considered respectively.

Case I : The first test function introduced is as follows

$$f(x, y, z, t) = E + A \cosh(\zeta) + B \cosh(\theta) + CP(\xi), \quad \zeta = R_1(K_1x + K_2y + K_3z + K_4t + K_0), \quad (7)$$

$$\theta = R_2(L_1x + L_2y + L_3z + L_4t + L_0), \quad \xi = R_3(J_1x + J_2y + J_3z + J_4t + J_0),$$

where $K_i, L_i, J_i (i = 0, 1, 2, 3, 4)$ and $A, B, C, E, R_1, R_2, R_3$ are undetermined constants and $P(\xi)$ is an arbitrary function of the variable ξ . By substituting Eq. (7) into Eq. (4), an equation in terms of the functions $\cosh(\zeta), \sinh(\zeta), \cosh(\theta), \sinh(\theta)$ and $\frac{\partial^n P}{\partial \xi^n} (n = 0, 1, 2, 3, 4)$ is produced:

$$F(\cosh(\zeta), \sinh(\zeta), \cosh(\theta), \sinh(\theta), P(\xi), \frac{\partial P}{\partial \xi}, \frac{\partial^2 P}{\partial \xi^2}, \frac{\partial^3 P}{\partial \xi^3}, \frac{\partial^4 P}{\partial \xi^4}) = 0. \quad (8)$$

From the linear independence of the above functions, we obtain a system of equations in $K_i, L_i, J_i (i = 0, 1, 2, 3, 4)$ and $A, B, C, E, R_1, R_2, R_3$, the solution of which gives the undetermined coefficients (we have omitted this lengthy process). $f(x, y, z, t)$ can be obtained as follows

$$f_1(x, y, z, t) = E + B \cosh[R_1(\frac{R_2 L_1}{R_1} x - \frac{L_2 R_2}{R_1} y - \frac{L_2 R_2 (R_2^2 L_1^3 + L_4)}{L_1 R_1} z + \frac{R_2 L_4}{R_1} t + K_0)] \quad (9)$$

$$+ B \cosh[R_2(L_1 x + L_2 y + \frac{L_2 (R_2^2 L_1^3 + L_4)}{L_1} z + L_4 t + L_0)] + CP_1(R_3 \xi), \quad \xi = J_2 y + \frac{J_2 (R_2^2 L_1^3 + L_4)}{L_1} z + J_0,$$

and

$$f_2(x, y, z, t) = E + A \cosh[R_1(K_1 x - \frac{K_1 (R_1^2 K_1^2 J_2 - J_3)}{J_2} t + K_0)] + B \cosh[R_2(L_2 y + \frac{L_2 J_3}{J_2} z + L_0)] \quad (10)$$

$$+ CP_2(R_3 \eta), \quad \eta = J_2 y + J_3 z + J_0,$$

where $A, B, C, E, K_0, L_0, J_0$ and the coefficients of the variables x, y, z, t are constants, $P_1(R_3 \xi)$ and $P_2(R_3 \eta)$ are functions of variables $\xi = J_2 y + \frac{J_2 (R_2^2 L_1^3 + L_4)}{L_1} z + J_0$ or $\eta = J_2 y + J_3 z + J_0$, respectively. Then the two solutions corresponding to Eq. (1) can be expressed as

$$u_1(x, y, z, t) = -2BL_1 R_2 \{ \sinh[R_1(\frac{R_2 L_1}{R_1} x - \frac{L_2 R_2}{R_1} y - \frac{L_2 R_2 (R_2^2 L_1^3 + L_4)}{L_1 R_1} z + \frac{R_2 L_4}{R_1} t + K_0)] \quad (11)$$

$$+ \sinh[R_2(L_1 x + L_2 y + \frac{L_2 (R_2^2 L_1^3 + L_4)}{L_1} z + L_4 t + L_0)] \} / \{ E + B \cosh[R_1(\frac{R_2 L_1}{R_1} x - \frac{L_2 R_2}{R_1} y$$

$$- \frac{L_2 R_2 (R_2^2 L_1^3 + L_4)}{L_1 R_1} z + \frac{R_2 L_4}{R_1} t + K_0)] + B \cosh[R_2(L_1 x + L_2 y + \frac{L_2 (R_2^2 L_1^3 + L_4)}{L_1} z + L_4 t + L_0)]$$

$$+ CP_1(R_3 (J_2 y + \frac{J_2 (R_2^2 L_1^3 + L_4)}{L_1} z + J_0)) \},$$

and

$$u_2(x, y, z, t) = -2AK_1R_1 \sinh[R_1(K_1x - \frac{K_1(R_1^2K_1^2J_2-J_3)}{J_2}t + K_0)]/\{E + A \cosh[R_1(K_1x - \frac{K_1(R_1^2K_1^2J_2-J_3)}{J_2}t + K_0)] + B \cosh[R_2(L_2y + \frac{L_2J_3}{J_2}z + L_0)] + CP_2(R_3(J_2y + J_3z + J_0))\}, \quad (12)$$

where $P_1(R_3(J_2y + \frac{J_2(R_2^2L_1^3+L_4)}{L_1}z + J_0))$ and $P_2(R_3(J_2y + J_3z + J_0))$ are two arbitrary functions.

Case II : The second test function introduced is as follows

$$f(x, y, z, t) = E + A(K_1x + K_2y + K_3z + K_4t + K_0)^2 + B(L_1x + L_2y + L_3z + L_4t + L_0)^2 + CP(\xi), \quad \xi = J_1x + J_2y + J_3z + J_4t + J_0, \quad (13)$$

where $K_i, L_i, J_i (i = 0, 1, 2, 3, 4)$ and A, B, C, E are undetermined constants, and $P(\xi)$ is an arbitrary function about ξ . Similarly, substituting Eq. (13) into Eq. (4), $f(x, y, z, t)$ can be obtained as follows

$$f_1(x, y, z, t) = E - \frac{BL_1^2}{K_1^2}(K_1x + K_2y + \frac{K_2L_4}{L_1}z + \frac{K_1L_4}{L_1}t + K_0)^2 + B(L_1x + L_2y + \frac{L_2L_4}{L_1}z + L_4t + L_0)^2 + P(J_2y + \frac{J_2L_4}{L_1}z + J_0), \quad (14)$$

and

$$f_2(x, y, z, t) = E + A(K_2y + \frac{K_2L_4}{L_1}z + K_0)^2 + B(L_1x + L_4t + L_0)^2 + P(J_2y + \frac{J_2L_4}{L_1}z + J_0), \quad (15)$$

where A, B, E, K_0, L_0, J_0 and the coefficients of the variables x, y, z, t are constants, $P(J_2y + \frac{J_2L_4}{L_1}z + J_0)$ is an arbitrary function. By substituting Eq. (14) and Eq. (15) into Eq. (2), the second set of solutions to Eq. (1) can be expressed as follows

$$u_3(x, y, z, t) = -4BL_1[(L_1x + L_2y + \frac{L_2L_4}{L_1}z + L_4t + L_0) - \frac{L_1}{K_1}(K_1x + K_2y + \frac{K_2L_4}{L_1}z + \frac{K_1L_4}{L_1}t + K_0)]/\{E - \frac{BL_1^2}{K_1^2}(K_1x + K_2y + \frac{K_2L_4}{L_1}z + \frac{K_1L_4}{L_1}t + K_0)^2 + B(L_1x + L_2y + \frac{L_2L_4}{L_1}z + L_4t + L_0)^2 + P(J_2y + \frac{J_2L_4}{L_1}z + J_0)\}, \quad (16)$$

and

$$u_4(x, y, z, t) = -\frac{4BL_1(L_1x + L_4t + L_0)}{E + A(K_2y + \frac{K_2L_4}{L_1}z + K_0)^2 + B(L_1x + L_4t + L_0)^2 + P(J_2y + \frac{J_2L_4}{L_1}z + J_0)}. \quad (17)$$

3 Analysis of dynamics for some solutions

The diversity in the evolution of solutions, as observed in the context of the previous statements, the presence of an arbitrary function and multiple parameters ‘explains’ the diversity. This complexity allows for a wide range of possible outcomes, which can be described as follows: due to the incorporation of an arbitrary function and a multitude of parameters, the solutions exhibit a rich variety of behaviors over time. This diversity is a direct consequence of the flexibility introduced by the arbitrary function and the interplay of multiple parameters, which can lead to a broad spectrum of dynamical responses.

The following illustrates some structures corresponding to solutions with digital images. By plotting these solutions against time or other relevant variables, we can observe how they evolve and interact under different conditions. The visualizations provide insights into the stability, periodicity, or chaotic tendencies of the solutions, offering a clear and intuitive understanding of their dynamical properties.

Case I : Three cases of $u_1(x, y, z, t)$ and their dynamical behavior in Eq. (11).

First, if setting $P_1(\xi) = \cosh(R_3(J_2y + \frac{J_2(R_2^2L_1^3+L_4)}{L_1}z + J_0))$ and $A = 2, B = 4, C = 20, E = 6, K_0 = 0, K_1 = 2, L_0 = 0, L_1 = 2, L_2 = 5, L_4 = 7, J_0 = 0, J_2 = 3, J_3 = 1, R_1 = 2i, R_2 = 4i, R_3 = 2$ in $u_1(x, y, z, t)$, we get the following chain soliton solution

$$u_5(x, y, z, t) = 32 \frac{\sin(8x-20y+1210z+28t)+\sin(8x+20y-1210z+28t)}{3+10 \cosh(3y-\frac{363}{2}z)+2 \cos(8x-20y+1210z+28t)+2 \cos(8x+20y-1210z+28t)}. \quad (18)$$

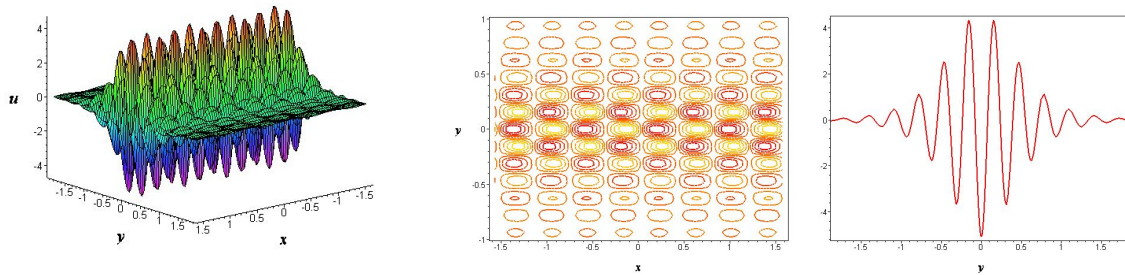


Fig. 1: Chain soliton solution $u_5(x, y, z, t)$ at $z = 0, t = 1$.

Dynamical analysis of $u_5(x, y, z, t)$:

(i) Wave interference structure: Numerator “ $\sin(8x - 20y + 1210z + 28t) + \sin(8x + 20y - 1210z + 28t)$ ” describes bidirectional wave propagation, while the denominator combines hyperbolic localization and phase-locked cosine terms.

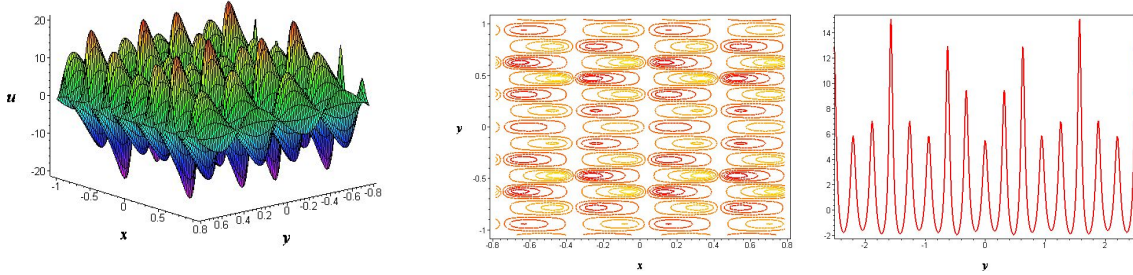
(ii) Anisotropic propagation: Phase velocities satisfy $v_x = 3.5$ (dominant), $v_y = \pm 1.4$ (shear), $v_z \approx \pm 0.023$ (confinement).

(iii) Stability mechanism: Temporal synchronization ($T = 0.224$) and spatial confinement by “ $10 \cosh(3y - 363z/2)$ ” stabilize the chain-soliton morphology.

To sum up, $u_5(x, y, z, t)$ embodies a 3D periodically localized wave structure formed by bidirectional wave interference within a hyperbolic potential. Its dynamics are characterized by:

Anisotropic propagation (dominant x -direction transport); **Spatiotemporal stability** from phase-locked interference; **High-dimensional confinement** enabling chain-soliton morphology.

Secondly, by defining $P_1(\xi) = \cosh(R_3(J_2y + \frac{J_2(R_2^2L_1^3+L_4)}{L_1}z + J_0))$, and the constants are given by: $A = 2, B = 4, C = 2, E = 12, K_0 = 0, K_1 = 2, L_0 = 0, L_1 = 2, L_2 = 5, L_4 = 7, J_0 = 0, J_2 = 3, J_3 = 1, R_1 = 2i, R_2 = 4i, R_3 = 2i$. By substituting these values into the function $u_1(x, y, z, t)$, the periodic

Fig. 2: Three-wave solution $u_6(x, y, z, t)$ at $z = 0, t = 1$.

three-wave solution can be obtained

$$u_6(x, y, z, t) = 32 \frac{\sin(8x-20y+1210z+28t) + \sin(8x+20y-1210z+28t)}{6 + 2 \cos(8x-20y+1210z+28t) + 2 \cos(8x+20y-1210z+28t) + \cos(6y-363z)}. \quad (19)$$

Dynamical behavior of $u_6(x, y, z, t)$:

The periodic three-wave solution achieves complex spatiotemporal structures through multi-phase interference, and its core dynamics can be summarized as **stable interference patterns formed** by directionally propagating periodic wave packets under nonlinear effects, **with amplitude and propagation direction modulated** by phase parameters. This rich dynamical behavior is vividly illustrated in the periodic motion of waves depicted in Fig. 2, which reveals a dynamic pattern evolving simultaneously along both the x - and y -axes. The wave amplitudes vary significantly across space and time, reflecting the intricate balance between constructive and destructive interferences mediated by phase coherence. Such a phenomenon bears a striking resemblance to fluctuating sea waves during a storm—where large swells and smaller ripples coexist and interact chaotically—yet here, the apparent complexity emerges from an underlying order: a coherent, nonlinearly sustained wavefront governed by deterministic equations. Thus, while the system exhibits visually turbulent and ever-changing profiles, it remains anchored in stable, repeating patterns that are sensitive to initial phase configurations, highlighting the profound interplay between nonlinearity, dispersion, and interference in shaping multidimensional wave dynamics.

Thirdly, when the function $P_1(\xi) = \cosh(R_3(J_2y + \frac{J_2(R_2^2L_1^3+L_4)}{L_1}z + J_0))$ and $r = R_1 = R_2 = R_3$ are set within the function $u_1(x, y, z, t)$ and the condition $E + 2B + C = 0$ holds with $r \rightarrow 0$, a lump solution can be derived as follows

$$u_7(x, y, z, t) = \frac{-32BL_1^3(4L_1x+4L_4t+K_0+2L_0)}{4B(2L_1^2x-2L_2L_1y-2L_2L_4z+2L_4L_1t+K_0L_1)^2+16B(L_1^2x+L_2L_1y+L_2L_4z+L_4L_1t+L_0L_1)^2+9C(J_2L_1y+J_2L_4z+J_0L_1)^2}. \quad (20)$$

Specifically, when the parameters are set as follows: $B = 4, C = 2, K_0 = 0, L_0 = 0, L_1 = 2, L_2 = 5, L_4 = 7, J_0 = 0, J_2 = 3, J_3 = 1$, the corresponding solution is illustrated in Fig.3. Upon examining

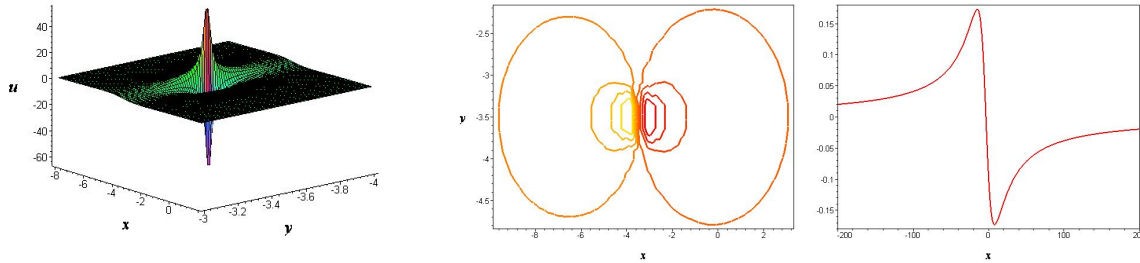


Fig. 3: Lump solution $u_7(x, y, z, t)$ at $z = 1, t = 1$.

Fig.3, it is evident that the solution represents a classic example of a rogue wave solution formed by rational functions.

Dynamical behavior of $u_7(x, y, z, t)$:

The dynamic behavior of the solution is mainly reflected in three aspects: locality, interaction mode, and spatiotemporal evolution.

(i) **Localization mechanism** : The Lump solution is constructed through a bilinear form, satisfying analyticity and rational localization conditions to ensure energy concentration in a limited region.

(ii) **Nature of interaction** : When the Lump wave interacts with line solitons, nonlinear effects lead to waveform fusion or fission, but the total energy is conserved.

(iii) **Spatiotemporal evolution** : Adjusting equation parameters (such as propagation speed and amplitude coefficient) can change the trajectory of Lump solutions.

In the analysis of the evolution of the function $u_1(x, y, z, t)$, the values of the parameters A, B, C and E play a crucial role in determining the nature of the solution. Specifically, the solution may exhibit singular behavior depending on these values. To ensure that the function $f(x, y, z, t)$ does not vanish, it is common practice to select a sufficiently large value for E and relatively small values for A, B and C .

Case II : Three cases of $u_2(x, y, z, t)$ and their digital images in Eq. (12).

Firstly, if setting $P_2(R_3\eta) = R_3(J_2y + J_3z + J_0)^2$ and $A = 2, B = 4, C = 2, E = 20, K_0 = 0, K_1 = 2, L_0 = 0, L_2 = 5, J_0 = 0, J_2 = 3, J_3 = 1, R_1 = 2i, R_2 = 4i, R_3 = 2$ in $u_1(x, y, z, t)$, we can obtain the following periodic lump solution

$$u_8(x, y, z, t) = \frac{8 \sin(4x + \frac{196}{3}t)}{10 + \cos(4x + \frac{196}{3}t) + 2 \cos(20y + \frac{20}{3}z) + 2(3y + z)^2}. \quad (21)$$

Dynamical behavior of $u_8(x, y, z, t)$:

(i) **Traveling wave in x, t** : The solution propagates periodically in the x -direction with velocity $v_x = -\frac{49}{3}$, maintaining its shape over time.

(ii) **Periodic in y, z** : Oscillations in y, z due to the cosine term, but modulated by the quadratic term to form a localized lump.

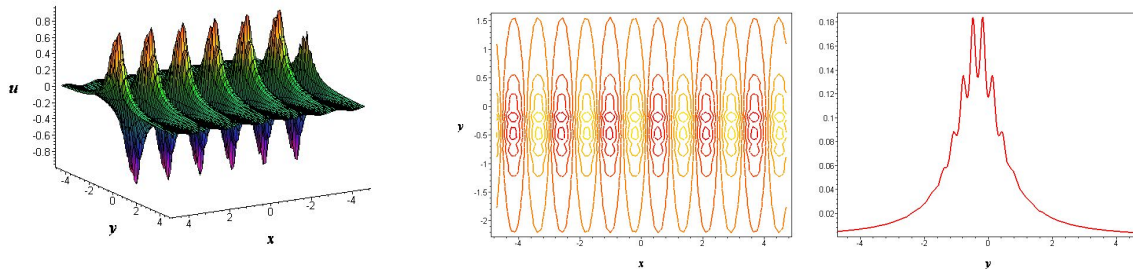


Fig. 4: Periodic lump solution $u_8(x, y, z, t)$ at $z = 1, t = 1$.

(iii) Stability: The denominator's non-vanishing nature (minimum value $10 - 1 - 2 + 0 = 7$ when $\cosines = -1$ and $3y + z = 0$) ensures the solution remains bounded and smooth, indicating stability.

Briefly describe: The periodic lump solution exhibits combined periodic traveling wave behavior in x, t and localized lump structure in y, z , characteristic of a stable, bounded nonlinear wave in the (3+1)-dimensional generalized shallow water wave system.

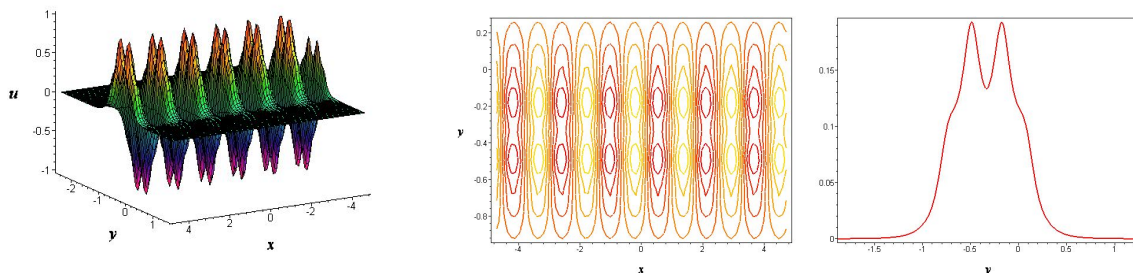


Fig. 5: Periodic lump solution $u_9(x, y, z, t)$ at $z = 1, t = 1$.

Secondly, by defining $P_2(R_3\eta) = R_3(J_2y + J_3z + J_0)^6$, and keeping all other parameters consistent with those of $u_8(x, y, z, t)$, one can derive another periodic lump solution

$$u_9(x, y, z, t) = \frac{8 \sin(4x + \frac{196}{3}t)}{12 + \cos(4x + \frac{196}{3}t) + 2 \cos(20y + \frac{20}{3}z) + 2(3y + z)^6}. \quad (22)$$

Comparison of Fig. 5 with Fig. 4 reveals that Fig. 5 displays a periodic lump solution that exhibits more pronounced oscillations in the y -direction.

Thirdly, when the function $P_2(R_3\eta) = \cosh(R_3(J_2y + J_3z + J_0))^2$ and $A = 2, B = 4, C = 2, E = 6, K_0 = 0, K_1 = 2, L_0 = 0, L_2 = 5, J_0 = 0, J_2 = 3, J_3 = 1, R_1 = 2i, R_2 = 4, R_3 = 2$ are set within the function $u_2(x, y, z, t)$, we get the following periodic soliton solution(breather solitary waves)

$$u_{10}(x, y, z, t) = \frac{8 \sin(4x + \frac{196}{3}t)}{3 + \cos(4x + \frac{196}{3}t) + 2 \cosh(20y + \frac{20}{3}z) + \cosh(6y + 2z)^2}. \quad (23)$$

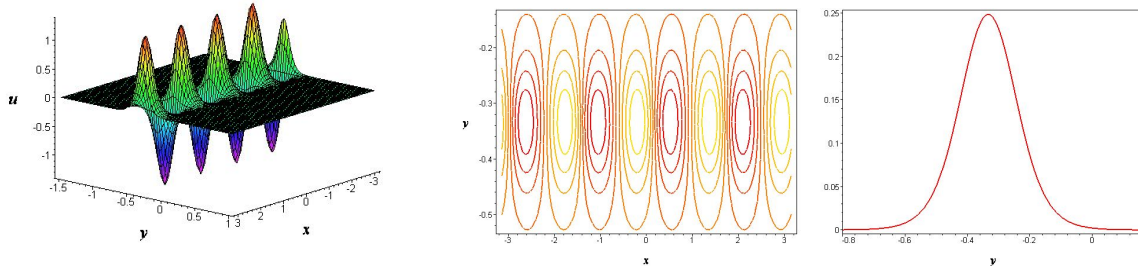


Fig. 6: Periodic soliton solution $u_{10}(x, y, z, t)$ at $z = 1, t = 1$.

Dynamical behavior of $u_{10}(x, y, z, t)$:

(i) **Temporal evolution:** The solution oscillates periodically in x and propagates in the $-x$ direction with velocity $v_x = -\frac{49}{3}$.

(ii) **Spatial structure:** Remains localized in the $y - z$ plane along $3y + z = C$, with no dispersion (shape preserved over time).

(iii) **Amplitude:** Modulated by the periodic sin term, leading to oscillating peaks in the x -direction while maintaining localization in $y - z$.

Briefly describe: The solution is a periodic lump soliton with periodic oscillations in the $x - t$ plane and exponential localization in the $y - z$ plane, propagating in the $-x$ direction without dispersion.

Similarly, an analysis can be conducted to explore the properties of the solutions $u_3(x, y, z, t)$ and $u_4(x, y, z, t)$. These solutions exhibit lump-shaped profiles, yet they experience oscillations attributed to the function $P(J_2y + \frac{J_2L_4}{L_1}z + J_0)$. However, a detailed analysis is not provided in this context.

4 Conclusion

This study has established two **new families of three-wave-hybrid solutions** for the (3+1)D generalized shallow water wave equation. The solutions incorporate arbitrary functions and multiple parameters, generating diverse wave structures including: **Chain-solitons, Periodic three-wave patterns, Rogue wave lumps**. These results significantly extend the known solution space and provide deeper insights into nonlinear wave dynamics. The dynamical characteristics of these partial solutions provide a more accurate representation of wave propagation on the water surface, thereby illuminates a multitude of nonlinear phenomena. This method is not only innovative but also versatile, as it can be extended to other multi-dimensional nonlinear equations, offering a broad scope for further exploration and application in the field of nonlinear dynamics.

Acknowledgments

The work was supported by the National Natural Science Foundation of China (No.12261053) and the Scientific Research Foundation of Department of Education of Yunnan Province (No.2022J0946).

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